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Oscillation of Higher-Order Nonlinear Difference Equations of Neutral Type

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Abstract—We shall establish some new criteria for the oscillation of all solutions of higher-order difference equations of the form

$$\Delta^m(x_n - x_{n-\tau}) + q_n f(x_{n-g}) = 0, \qquad m \geq 1.$$

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1. INTRODUCTION

In this paper, we are concerned with the oscillatory behavior of higher-order difference equations of the form

$$\Delta^m(x_n - x_{n-\tau}) + q_n f(x_{n-g}) = 0, \tag{1}$$

where Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$, and for $i \geq 1$, Δ^i is the i^{th}-order forward difference operator $\Delta^i x_n = \Delta(\Delta^{i-1} x_n)$. The following conditions are always assumed to hold:

(a) $\{q_n\}$ is a real sequence with $q_n \geq 0$, eventually, and $n \in \mathbb{N} = \{0, 1, 2, \dots\}$,
(b) $f : \mathbb{R} \to \mathbb{R}$ is continuous with $uf(u) > 0$ and nondecreasing for $u \neq 0$, and satisfies

$$-f(-xy) \geq f(xy) \geq f(x)f(y), \qquad \text{for } xy > 0,$$

(c) g, τ are positive integers.

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Let $\sigma = \max\{\tau, g\}$ and N_0 be a fixed nonnegative integer. By a solution of (1), we mean a real sequence $\{x_n\}$ which is defined for all $n \geq N_0 \geq \sigma$, and satisfies (1) for $n \geq N_0$. A solution $\{x_n\}$ of (1) is said to be nonoscillatory if all terms x_n are eventually of one sign. Otherwise the solution $\{x_n\}$ is called oscillatory. In this paper, we shall be concerned only with the nontrivial solutions of (1).

The problem of finding sufficient conditions which ensure that all solutions of certain classes of difference equations of neutral type are oscillatory has been studied by a number of authors. Here, we refer to [1–6] and the references therein.

Equation (1) is a discrete analog of the neural differential equation

$$(x(t) - x[t-\tau])^{(m)} + q(t)f(x[t-g]) = 0, \tag{1*}$$

where $q : [t_0, \infty) \to [0, \infty)$ is continuous, f satisfies Condition (b), and τ and g are real numbers.

The oscillatory behavior of (1)* with $m = 1$ and $f(x) = x$ has been studied by many authors and as recent contributions, we refer to [7–10] and the references therein. It is shown in [7] that $\int^\infty q(s)\,ds = \infty$ is an essential condition for the oscillation of (1)* with $m = 1$ and $f(x) = x$. In [10] they proceeded further and in the case $\int^\infty q(s)\,ds < \infty$, they proved that the condition

$$\int^\infty sq(s) \int_s^\infty q(u)\,du\,ds = \infty$$

is sufficient for the oscillation of (1)* with $m = 1$ and $f(x) = x$.

It seems that only little is known regarding the oscillation of nonlinear equation (1) with $m \geq 1$. Therefore, the purpose of this paper is to establish some new criteria for the oscillation of (1). The results of this paper are applicable to superlinear ($\int^{\pm\infty} \frac{du}{f(u)} < \infty$), linear ($f(x) = x$) and sublinear ($\int_{\pm 0} \frac{du}{f(u)} < \infty$) equations of type (1). We also note that the results obtained for (1) with $m = 1$ and $f(x) = x$ are the discrete analogs of the above mentioned results for (1)*, with $m = 1$ and $f(x) = x$.

2. MAIN RESULTS

We shall need the following two lemmas which are given in [1].

LEMMA 1. *Let x_n be defined for $n \geq n_0$ and $x_n > 0$ with $\Delta^m x_n$ eventually of one sign. Then, there exists an integer j, $0 \leq j \leq m$ with $(m+j)$ odd for $\Delta^m x_n \leq 0$ and $(m+j)$ even for $\Delta^m x_n \geq 0$ and an integer $N \geq n_0$, such that for all $n \geq N$*

$$j \leq m-1 \text{ implies } (-1)^{j+i}\Delta^i x_n > 0, \qquad j \leq i \leq m-1$$

and

$$j \geq 1 \text{ implies } \Delta^i x_n > 0, \qquad 1 \leq i \leq m-1.$$

LEMMA 2. *Let x_n be defined for $n \geq n_0$ and $x_n > 0$ with $\Delta^m x_n \leq 0$ eventually. If x_n is increasing, then there exists a sufficiently large $N \geq n_0$ such that*

$$x_n \geq \frac{2^{2-2m}}{(m-1)!} n^{(m-1)} \Delta^{m-1} x_n, \qquad \text{for all } n \geq 2^{m-1}N.$$

THEOREM 1. *Let m be odd. If*

$$\sum^\infty q_k < \infty \tag{2}$$

and

$$\sum^\infty q_n f(nQ_n) = \infty, \tag{3}$$

where

$$Q_n = \sum_{k=n}^{\infty} q_n, \tag{4}$$

then (1) is oscillatory.

PROOF. Let $\{x_n\}$ be an eventually positive solution of (1), say $x_n > 0$, $x_{n-\tau} > 0$, and $x_{n-g} > 0$ for $n \geq n_1 \geq n_0$. Set

$$z_n = x_n - x_{n-\tau}. \tag{5}$$

Then (1) becomes

$$\Delta^m z_n = -q_n f(x_{n-g}) \leq 0, \qquad n \geq n_1. \tag{6}$$

Thus, $\Delta^i z_n$ are eventually of one sign, $i = 0, 1, \ldots, m$, and there are four possible cases to consider:

(A) $z_n > 0$, $\Delta z_n > 0$,
(B) $z_n > 0$, $\Delta z_n < 0$,
(C) $z_n < 0$, $\Delta z_n > 0$, and
(D) $z_n < 0$, $\Delta z_n < 0$, eventually.

CASE A. Assume $z_n > 0$ and $\Delta z_n > 0$ for $n \geq n_1$. From (5) we see that $x_n \geq z_n$, and hence, there exist an integer $n_2 \geq n_1$ and a positive constant c such that

$$x_{n-g} \geq z_{n-g} \geq c, \qquad \text{for } n \geq n_2.$$

Thus,

$$\Delta^m z_n \leq -f(c) q_n, \qquad \text{for } n \geq n_2. \tag{7}$$

It is easy to check $\Delta^{m-1} z_n > 0$ for $n \geq n_2$. Summing both sides of (7) from $n \geq n_2$ to $N \geq n$ and letting $N \to \infty$, we obtain

$$\Delta^{m-1} z_n \geq f(c) \sum_{k=n}^{\infty} q_k = f(c) Q_n, \qquad n \geq n_2. \tag{8}$$

Next, applying Lemma 2, there exists an integer $n_3 \geq 2^{m-1} n_2$ so large that

$$z_n \geq a n^{(m-1)} \Delta^{m-1} z_n, \qquad \text{for } n \geq n_3, \tag{9}$$

where $\alpha = 2^{2-2m}/(m-1)!$. Using (9) in (8), we have

$$x_n \geq z_n \geq \alpha_1 n^{m-1} Q_n, \qquad n \geq n_3,$$

where $\alpha_1 = \alpha f(c)$. Now, there exists an integer $n_4 \geq n_3$ such that

$$x_{n-g} \geq \alpha_1 (n-g)^{m-1} Q_{n-g} \geq \alpha_1 (n-g)^{m-1} Q_n, \qquad n \geq n_4. \tag{10}$$

Using (6) and (10) in (6), we obtain

$$\Delta^m z_n \leq -\alpha_2 q_n f\left((n-g)^{m-1} Q_n\right), \qquad n \geq n_4, \tag{11}$$

where $\alpha_2 = f(\alpha_1)$.

Summing (11) from n_4 to $n - 1 \geq n_4$, we get

$$\Delta^{m-1} z_n \leq \Delta^{m-1} z_{n_4} - \alpha_2 \sum_{j=n_4}^{n-1} q_j f\left((j-g)^{m-1} Q_j\right),$$

which in view of (3) leads to

$$\Delta^{m-1} z_n \to -\infty, \qquad \text{as } n \to \infty,$$

a contradiction.

CASE B. Assume $z_n > 0$ and $\Delta z_n < 0$ eventually. By Lemma 1, we see that $\Delta^{m-1} z_n > 0$ eventually. In this case, we have $x_n > x_{n-\tau}$. Hence, there exist a constant $b > 0$ and $n_2 \geq n_1$ such that

$$x_{n-g} \geq b, \qquad \text{for all } n \geq n_2.$$

Then, from (6), it follows that

$$\Delta^m z_n \leq -f(b) q_n, \qquad \text{for } n \geq n_2,$$

and hence,

$$\Delta^{m-1} z_n \geq f(b) Q_n, \qquad \text{for } n \geq n_2. \tag{12}$$

By Taylor's formula (see [1]), we see that for $s-1 \geq k \geq j \geq n_2$

$$z_k = \sum_{i=0}^{m-2} \frac{(s+i-1-k)^{(i)}}{i!} (-1)^i \Delta^i z_s + \sum_{j=k}^{s-1} \frac{(j+m-2-k)^{(m-2)}}{(m-2)!} (-1)^{m-1} \Delta^{m-1} z_j.$$

Since m is odd, and $\Delta^{m-1} z_n$ is decreasing, one can easily see that

$$z_k \geq \left(\sum_{j=k}^{s-1} \frac{(j+m-2-k)^{(m-2)}}{(m-2)!} \right) \Delta^{m-1} z_{s-1}.$$

Replacing k with $n-\tau$ and s with $n+1$, we have

$$z_{n-\tau} \geq \left(\sum_{j=n-\tau}^{n} \frac{(j+m-2-n+\tau)^{(m-2)}}{(m-2)!} \right) \Delta^{m-1} z_n$$

or

$$z_n \geq \beta \Delta^{m-1} z_{n+\tau}, \qquad \text{for } n \geq n_3 \geq n_2, \tag{13}$$

where

$$\beta = \sum_{j=n}^{n+\tau} \frac{(j+m-2-n)^{(m-2)}}{(m-2)!}.$$

Using (13) in (12), we obtain

$$z_n \geq \beta_1 Q_{n+\tau}, \qquad n \geq n_3, \tag{14}$$

where $\beta_1 = \beta f(b)$. From (5), we have

$$x_n \geq \beta_1 Q_{n+h} + x_{n-\tau}, \qquad n \geq n_3.$$

Let an integer N be such that $n_3 + (N-1)\tau \leq n \leq n_3 + N\tau$. Then, we have

$$x_n \geq \beta_1 \left[Q_{n+\tau} + Q_n + \cdots + Q_{n-(N-2)\tau} \right] + x_{n-N\tau} \geq \beta_1 (N-1) Q_n,$$

which together with (6) yields

$$\Delta^m z_n \leq -A_n, \tag{15}$$

where

$$A_n = f\left(\frac{\beta_1(N-1)}{n}\right) q_n f(nQ_n).$$

By noting that $n/N \to \tau$ as $n \to \infty$, we find

$$\frac{A_n}{q_n f(nQ_n)} = f\left(\beta_1\left(\frac{N-1}{n}\right)\right) \to f\left(\frac{\beta_1}{\tau}\right), \qquad \text{as } n \to \infty$$

and by (3), we have

$$\sum^{\infty} A_n = \infty. \tag{16}$$

Thus, (15) and (16) yield

$$\Delta^{m-1} z_n \to -\infty, \qquad \text{as } n \to \infty,$$

a contradiction.

CASE C. Assume $z_n < 0$ and $\Delta z_n > 0$ eventually. Then,

$$0 < v_n = -z_n = x_{n-\tau} - x_n,$$

and hence, (6) become

$$\Delta^m v_n = q_n f(x_{n-g}), \qquad \text{eventually.}$$

Since m is odd, by Lemma 1, we must have $\Delta v_n > 0$ eventually contradiciting the assumption.

CASE D. Assume $z_n < 0$ and $\Delta z_n < 0$ for $n \geq n_1$. Since z_n is decreasing, there exist a constant $a > 0$ and a $T \geq n_1$ such that

$$z_n < -a, \qquad \text{for } n \geq N.$$

Therefore,

$$x_N = z_N + x_{N-\tau} < -a + x_{N-\tau}$$

and it follows that

$$x_{N+j\tau} < -a(j+1) + x_{N-\tau} \to -\infty, \qquad \text{as } j \to \infty,$$

which contradicts $x_n > 0$ eventually. This completes the proof.

REMARK 1. When $\tau = 1$, (1) is reduced to an even order equation

$$\Delta^{m+1} x_n + q_n f(x_{n-g}) = 0. \tag{17}$$

Applying Theorem 1 for (17), we obtain the following new criterion for the oscillation of (17).

COROLLARY 1. *If*

$$\sum^{\infty} q_n f((n-g)^m Q_n) = \infty, \tag{18}$$

where Q_n is defined as in (4), then (17) is oscillatory.

PROOF. The proof is contained in the proof of Theorem 1, Case A.

Now we introduce the following notation:

$$g^* = \begin{cases} g, & \text{if } m \text{ is odd}, \\ g - \tau > 0, & \text{if } m \text{ if even}. \end{cases} \tag{19}$$

Next we shall prove the following comparison result.

THEOREM 2. *Suppose that the first-order equation*

$$\Delta w_n + f\left(\frac{2^{2-2m}}{(m-1)!}\right) q_n f\left((n-g)^{m-1}\right) f(w_{n-g}) = 0 \tag{20}$$

is oscillatory, and all bounded solutions of the equation

$$\Delta^m y_n + (-1)^{m+1} q_n f(y_{n-g^*}) = 0 \tag{21}$$

are oscillatory. Then, (1) is oscillatory.

PROOF. Let $\{x_n\}$ be an eventually positive solution of (1), and let z_n be defined as in (5). As in the proof of Theorem 1, the four Cases A–D need to be considered.

CASE A. Assume $z_n > 0$ and $\Delta z_n > 0$ for $n \geq n_1 \geq n_0$. Proceeding as in the proof of Theorem 1 Case A, there exists an integer $n_2 \geq 2^{m-1} n_1$ such that $\Delta^{m-1} z_n > 0$ and (9) holds for $n \geq n_2$. From the fact that $x_n \geq z_n$ and Condition (c), there exists an integer $n_3 \geq n_2$ such that

$$x_{n-g} \geq z_{n-g} \geq \alpha (n-g)^{m-1} \Delta^{m-1} z_{n-g}, \qquad \text{for } n \geq n_3, \tag{22}$$

where $\alpha = 2^{2-2m}/(m-1)!$. Using Condition (b) and (22) in (6) and letting $v_n = \Delta^{m-1} z_n, n \geq n_3$, we have

$$\Delta v_n + f(\alpha) f\left((n-g)^{m-1}\right) q_n f(v_{n-g}) \leq 0, \qquad \text{for } n \geq n_3. \tag{23}$$

Summing both sides of (23) from $n \geq n_3$ to N and letting $N \to \infty$, we obtain

$$v_n \geq f(\alpha) \sum_{k=n}^{\infty} f\left((k-g)^{m-1}\right) q_k f(v_{k-g}). \tag{24}$$

But, by the discrete analog of a result in [11] and Theorem 1 in [12], if (24) has an eventually positive solution v_n, then the corresponding equation

$$w_n = f(\alpha) \sum_{k-n}^{\infty} f\left((k-g)^{m-1}\right) q_k f(w_{n-g}) \tag{25}$$

also has an eventually positive solution w_n. It follows then that equation (20) has the eventually positive solution w_n. This contradicts the hypothesis that (20) is oscillatory.

CASE B. Assume $z_n > 0$ and $\Delta z_n < 0$ for $n \geq n_1 \geq n_0$. This is the case when m is odd. By Lemma 1 and Lemma 6 in [11], there exists an integer $n_2 \geq n_1$ such that

$$(-1)^i \Delta^i z_n > 0, \qquad \text{for } i = 0, 1, \dots, m-1 \text{ and } n \geq n_2. \tag{26}$$

Summing both sides of (6) from $n \geq n_2$ to N repeatedly m-times, using (26) and the fact that $x_n \geq z_n$ for $n \geq n_2$ and letting $N \to \infty$, we have

$$z_n \geq \sum_{j_1=n}^{\infty} \sum_{j_2=j_1}^{\infty} \cdots \sum_{j_m=j_{m-1}}^{\infty} q_{j_m} f(z_{j_{m-g}}). \tag{27}$$

The remainder of the proof is similar to that of Case A given above.

CASE C. Assume $z_n < 0$ and $\Delta z_n > 0$ for $n \geq n_1 \geq n_0$. This is the case when m is even. Set

$$0 < y_n = -z_n = x_{n-\tau} - x_n.$$

Then, (1) becomes

$$\Delta^m y_n = q_n f(x_{n-g}) \tag{28}$$

and

$$x_{n-\tau} \geq y_n \qquad \text{or} \qquad x_n \geq y_{n+\tau}, \qquad \text{for } n \geq n_1. \tag{29}$$

Since m is even and $\Delta y_n < 0$ for $n \geq n_1$, there exists $n_2 \geq n_1$ such that

$$(-1)^i \Delta^i y_n > 0, \qquad \text{for } i = 0, 1, \dots, m-1 \text{ and } n \geq n_2. \tag{30}$$

Summing both sides of (28) from $n \geq n_2$ to N repeatedly m-times and using (29) and (30) and letting $N \to \infty$, we have

$$y_n \geq \sum_{j_1=n}^{\infty} \sum_{j_2=j_1}^{\infty} \cdots \sum_{j_m=j_{m-1}}^{\infty} q_{j_m} f\left(y_{j_m-(g-\tau)}\right).$$

The rest of the proof is similar to that of Case A given above.

CASE D. Assume $z_n < 0$ and $\Delta z_n < 0$ for $n \geq n_1 \geq n_0$. The proof is exactly the same as in the proof of Theorem 1, Case D. This completes the proof.

As an application of Theorem 2, we consider the equation

$$\Delta^m (x_n - x_{n-\tau}) + q x_{n-g} = 0, \tag{31}$$

where q is a real number and τ and g are positive integers. We see that (31) is oscillatory if

$$q > \frac{m^m g^g}{(m+g)^{m+g}}, \qquad g \geq 1. \tag{32}$$

REMARK 2. The results of this paper are presented in a form which is essentially new. Also, we see that the results of the paper are valid for equations of the form

$$\Delta^m (x_n - c x_{n-\tau}) + q_n f(x_{n-g}) = 0,$$

where c is a real number and $0 \leq c < 1$. Details are left to the reader.

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